Math 3360 - Foundations of Algebra I
Tutorial Assignment Sheet 1

1. Determine whether or not the following systems are groups:

- **(a)** The set of rational numbers, $\mathbb{Q}$, with operation $\star$ given by $a \star b = (a+b)/2$.
- **(b)** The set of non-zero rational numbers, $\mathbb{Q}^*$, with operation $\star$ given by $a \star b = (ab)/2$.
- **(c)** The set of integers, $\mathbb{Z}$, under subtraction.
- **(d)** The set $S := \{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\}$ under multiplication.

2. Determine whether or not the following sets are groups under addition and/or under multiplication:

- **(a)** The set $O(2,\mathbb{R}) := \{U \in GL(2,\mathbb{R}) \mid U^{-1} = U^t\}$.
- **(b)** The set $S := \{A \in M(2,\mathbb{R}) \mid A = \left[\begin{smallmatrix} a & a \\ a & a \end{smallmatrix}\right]\}$.

3. Let G be a group.

- **(a)** For a, $b \in G$, show that $(ab)^{-1} = b^{-1}a^{-1}$.
- **(b)** If $(ab)^2 = a^2b^2$ for all $a, b \in G$, prove that G is abelian.

4. Let S_3 denote the symmetric group of degree 3. Prove that its centre $Z(S_3) = \{(1)\}$.

5. Let $\rho = (1234)(234)$ be a permutation in S_6. What is ρ^{1001}?

6. Prove that if G is a finite group of prime order, then G is cyclic.

7. Let G be a group and H be a subgroup of G. Show that

$$N_G(H) := \{g \in G \mid gH = Hg\}$$

is a subgroup of G.

Note: $N_G(H)$ is called the normalizer of H in G.

8. Show that every group with 4 elements must be abelian.